June 25th, 2024

Review.

1) Integration by parts

$$\int u\,dv = uv - \int v\,du$$

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du.$$

a) $\int \ln 2x\,dx$ $\qquad (\ln u)' = \frac{u'}{u}$

$$\begin{cases} u = \ln 2x \\ dv = dx \end{cases} \Rightarrow \begin{cases} du = \frac{2}{2x}\,dx = \frac{dx}{x} \\ v = x \end{cases}$$

$$\int \ln 2x\,dx = x\ln 2x - \int x\cdot\frac{dx}{x}$$

$$= x\ln 2x - \int dx$$

$$= x\ln 2x - x + C.$$

b) $\int_0^1 x\,3^x\,dx$ $\qquad$ How to choose u $\Rightarrow$ du simple.

$$\begin{cases} u = x \\ dv = 3^x dx \end{cases} \Rightarrow \begin{cases} du = dx \\ v = \frac{3^x}{\ln 3} \end{cases}$$

$$\int_0^1 x e^x\,dx = x\cdot\frac{3^x}{\ln 3}\Big|_0^1 - \int_0^1 \frac{3^x}{\ln 3}\,dx$$

$$= \frac{3}{\ln 3} - \frac{1}{\ln 3}\,\frac{3^x}{\ln 3}\Big|_0^1$$

$$= \frac{3}{\ln 3} - \frac{1}{\ln^2 3}(3 - 1)$$

$$= \frac{2}{\ln 3} - \frac{2}{\ln^2 3}.$$

c) $\int e^{\sqrt{x}}\,dx.$

If $u = e^{\sqrt{x}} \Rightarrow du = e^{\sqrt{x}} \frac{1}{2\sqrt{x}}\,dx$ complicated

Substitution $u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}\,dx$

$\Rightarrow dx = 2\sqrt{x}\,du = 2u\,du.$

$\rightarrow \int e^{\sqrt{x}}\,dx = \int e^{u}(2u)\,du.$

Integration by parts:

$\begin{cases} t = 2u \\ dv = e^{u}\,du \end{cases} \rightarrow \begin{cases} dt = 2\,du \\ v = \frac{e^{u}}{\ln e} = e^{u} \end{cases}$ $\ln e = 1$

$\rightarrow \int e^{u}\,2u\,du = 2ue^{u} - 2\int e^{u}\,du$

$= 2ue^{u} - 2e^{u} + C$

$= 2\sqrt{x}\,e^{\sqrt{x}} - 2e^{\sqrt{x}} + C.$ ▨

d) $\int \cos(\ln x)\,dx.$

If $u = \cos(\ln x) \Rightarrow du = -\sin(\ln x)\frac{1}{x}\,dx$

$\Rightarrow$ complicated.

Substitution $u = \ln x \Rightarrow du = \frac{1}{x}\,dx$

$\Rightarrow dx = x\,du.$

$\int \cos(\ln x)\,dx = \int \cos u\,(x)\,du$

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$$x$$
$$dx = x\,du.$$

$$\int \cos(\ln x)\,dx = \int \cos u\,(x)\,du$$

But we have $u = \ln x \Rightarrow e^u = e^{\ln x} = x$

$$\int \cos(\ln x)\,dx = \int \cos u \cdot e^u\,du.$$

Integration by parts

$$\begin{cases} t = \cos u \\ dv = e^u\,du \end{cases} \Rightarrow \begin{cases} dt = -\sin u\,du \\ v = e^u \end{cases}$$

$$\int \cos u\,e^u\,du = e^u \cos u - \int(-\sin u)\,e^u\,du.$$

Integration by parts (one more time.)

$$\begin{cases} t = -\sin u \\ dv = e^u\,du \end{cases} \Rightarrow \begin{cases} dt = -\cos u\,du \\ v = e^u \end{cases}$$

$$\int \cos u\,e^u\,du = e^u \cos u - \left(-\sin u\,e^u - \int(-\cos u)\,e^u\,du\right)$$

$$\Rightarrow \int \cos u\,e^u\,du = e^u \cos u + \sin u\,e^u - \int \cos u\,e^u\,du.$$

$$\Rightarrow 2\int \cos u\,e^u\,du = e^u \cos u + e^u \sin u.$$

$$\Rightarrow \int \cos u\,e^u\,du = \frac{e^u \cos u + e^u \sin u}{2} + C.$$

$$= \frac{e^{\ln x}\cos(\ln x) + e^{\ln x}\sin(\ln x)}{2} + C$$

$$= \frac{x\cos(\ln x) + x\sin(\ln x)}{2} + C.$$

2) Trigonometric integrals.

$\sin^2$ ...

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2) Trigonometric integrals.

$$\sin^2 x + \cos^2 x = 1$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\tan^2 x + 1 = \sec^2 x.$$

$$d(\tan x) = \sec^2 x \, dx.$$

$$d(\sec x) = \sec x . \tan x \, dx.$$

$$\int \tan x \, dx = \ln|\sec x| + C$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C.$$

Example:

a) $\int \sin^3 x \cos^2 x \, dx$

$$= \int \underline{\sin^2 x} \sin x \cos^2 x \, dx$$

$$= \int (1 - \cos^2 x)\cos^2 x \, \sin x \, dx.$$

$$= \int (\cos^2 x - \cos^4 x) \underline{\sin x \, dx}.$$

Substitution: ~~$u = \sin x \Rightarrow du = \cos x \, dx.$~~ (not correct)

$u = \cos x \Rightarrow du = -\sin x \, dx \Rightarrow dx = \dfrac{du}{-\sin x}$

$$= \int (u^2 - u^4) \sin x \frac{du}{-\sin x}$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C.$$

b) $\int \tan x \sec x \, dx$

b) $\int \tan x \sec^3 x \, dx$

$= \int \tan x \sec x \sec^2 x \, dx$

$u = \sec x$

$\Rightarrow du = \tan x \sec$

$= \int \sec^2 x \cdot \tan x \sec x \, dx.$

Substitution: $u = \sec x \Rightarrow du = \tan x \cdot \sec x \, dx$

$= \int u^2 \, du$

$= \frac{u^3}{3} + C = \frac{\sec^3 x}{3} + C.$

c) $\int \sin^2 x \sin 2x \, dx$

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$= \frac{1}{2}\int (1 - \cos 2x) \sin 2x \, dx$

$= \frac{1}{2}\int \sin 2x \, dx - \frac{1}{2}\int \cos 2x \sin 2x \, dx$

Substitution $u = \cos 2x \Rightarrow du = -2\sin 2x \, dx$

$\Rightarrow dx = \frac{-1}{2\sin 2x} du.$

$= -\frac{1}{2} \frac{\cos 2x}{2} - \frac{1}{2}\int u \sin 2x \left(\frac{-1}{2\sin 2x}\right) du$

$= -\frac{1}{4}\cos 2x + \frac{1}{4}\int u \, du$

$$= -\frac{1}{4}\cos 2x + \frac{1}{4}\int u\,du$$

$$= -\frac{1}{4}\cos 2x + \frac{1}{4}\frac{u^2}{2} + c$$

$$= -\frac{1}{4}\cos 2x + \frac{1}{8}\cos^2 2x + C.$$

Another method: $\sin 2x = 2\sin x\cos x.$

$$\int \sin^2 x \sin 2x\,dx = 2\int \sin^2 x \sin x \cos x\,dx$$

$$= 2\int \sin^3 x \cos x\,dx$$

$$u = \sin x \Rightarrow du = \cos x\,dx$$

$$= 2\int u^3\,du$$

$$= 2\frac{u^4}{4} + c$$

$$= \frac{1}{2}\sin^4 x + C.$$

3/ Trigonometric substitution.

a) $\sqrt{a^2-x^2}$ $x = a\sin\theta$, $1-\sin^2\theta = \cos^2\theta$

b) $\sqrt{a^2+x^2}$ $x = a\tan\theta$, $1+\tan^2\theta = \sec^2\theta$

c) $\sqrt{x^2-a^2}$ $x = a\sec\theta$, $\sec^2\theta - 1 = \tan^2\theta$

Example:

a) $\int \frac{x^3}{\sqrt{1-x^2}}\,dx$ $x = \sin\theta \Rightarrow dx = \cos\theta\,d\theta$

$= \int \frac{\sin^3\theta}{\sqrt{1-\sin^2\theta}} \cos\theta\,d\theta$ $\sin^2\theta + \cos^2\theta = 1$

$= \int \frac{\sin^3\theta}{\cos\theta} \cos\theta\,d\theta$ $\Rightarrow \cos^2\theta = 1-\sin^2\theta$

$= \int \sin^3\theta\,d\theta$

$= \int \sin^2\theta\, \boxed{\sin\theta\,d\theta}$ $\int \sin\theta\,dx = -\cos\theta$

$= \int (1-\cos^2\theta)\sin\theta\,d\theta$

Substitution $u = \cos\theta \Rightarrow du = -\sin\theta\,d\theta$

$= -\int (1-u^2)\,du$

b) $\int_0^2 \frac{dt}{\sqrt{4+t^2}}$ $t = 2\tan\theta \Rightarrow dt = 2\sec^2\theta\,d\theta$

$\begin{cases} t=0 \\ t=2 \end{cases} \Rightarrow \begin{cases} \theta = 0 \\ \theta = \frac{\pi}{4} \end{cases}$

$\int^{\pi/4} \frac{2\sec^2\theta\,d\theta}{\sqrt{4+\tan^2\theta}} = \int_0^{\pi/4} \frac{2\sec^2\theta\,d\theta}{2\sec\theta}$

$= \int_0^{\pi/4} \sec\theta\,d\theta$

$= \ln|\sec\theta + \tan\theta| \Big|_0^{\pi/4}$

c) $\int x^2 \sqrt{3+2x-x^2}\,dx$.

Completing the square

$= 3 - (x^2 - 2x + 1 - 1)$

$+ \tan\theta \Big|_0^{\pi/4}$

c) $\int x^2 \sqrt{3 + 2x - x^2}\, dx$.

Completing the square

$$3 + 2x - x^2 = 3 - (x^2 - 2x)$$
$$= 3 - (x^2 - 2x + 1 - 1)$$
$$= 4 - (x^2 - 2x + 1)$$
$$= 4 - (x-1)^2$$

$$\int x^2 \sqrt{4 - (x-1)^2}\, dx$$

$y = x - 1 = 2\sin\theta$ –

$\Rightarrow x = 2\sin\theta + 1 \Rightarrow dx = 2\cos\theta\, d\theta$

$$= \int (2\sin\theta + 1)^2 \sqrt{4 - (2\sin\theta)^2}\; 2\cos\theta\, d\theta$$
$$= \int (2\sin\theta + 1)^2\; 2\cos\theta\quad 2.\cos\theta\, d\theta$$
$$= 4\int (2\sin\theta + 1)^2 \cos^2\theta\, d\theta$$
$$= 4\int (4\sin^2\theta + 4\sin\theta + 1)\cos^2\theta\, d\theta$$
$$= 4\Big[\int 4\sin^2\theta\cos^2\theta\, d\theta + \int 4\sin\theta\cos^2\theta\, d\theta + \int \cos^2\theta\, d\theta\Big]$$

d) $\int \sqrt{x^2 + 3x}\, dx$.

Completing the square. $(A+B)^2 = A^2 + 2AB + B^2$

$$x^2 + 3x = x^2 + 2x\tfrac{3}{2} + \tfrac{9}{4} - \tfrac{9}{4}$$
$$A^2 + 2A.B + B^2$$
$$= \left(x + \tfrac{3}{2}\right)^2 - \tfrac{9}{4}$$

$$= \int \sqrt{\left(x+\tfrac{3}{2}\right)^2 - \left(\tfrac{3}{2}\right)^2}\, dx$$

$\left(x + \frac{3}{2} = y = \frac{3}{2}\sec\theta\right)$

$\Rightarrow$ Substitution $x = \frac{3}{2}\sec\theta - \frac{3}{2}$.

$= \int \sqrt{\ldots} \ldots \frac{3}{2}\sec\theta\tan\theta\, d\theta$

$$\Rightarrow dx = \frac{3}{2}\sec\theta\tan\theta\, d\theta.$$

$$= \int \sqrt{\frac{3^2}{2^2}\sec^2\theta - \frac{3^2}{2^2}}\ \frac{3}{2}\sec\theta\tan\theta\, d\theta.$$

$$= \frac{3}{2}\int \frac{3}{2}\tan\theta\ \sec\theta\ \tan\theta\, d\theta$$

$$= \frac{9}{4}\int \tan^2\theta\ \sec\theta\ d\theta.$$

$$= \frac{9}{4}\int (\sec^2\theta - 1)\sec\theta\, d\theta$$

$$= \frac{9}{4}\int (\sec^3\theta - \sec\theta)\, d\theta$$

$$= \frac{9}{4}\int \sec^3\theta\, d\theta - \frac{9}{4}\int \sec\theta\, d\theta$$

We have $\frac{9}{4}\int \sec\theta\, d\theta = \frac{9}{4}\ln|\sec\theta + \tan\theta| + C.$

We are left with $\frac{9}{4}\int \sec^3\theta\, d\theta$ (complicated).

$$= \frac{9}{4}\int \frac{1}{\cos^3\theta}\, d\theta$$

4) Integration of rational functions by partial fractions

Example.

a) $\displaystyle\int \frac{1}{(x-3)(x+5)}\, dx$

$$\frac{1}{(x-3)(x+5)} = \frac{A}{x-3} + \frac{B}{x+5}$$

$$\Rightarrow 1 = A(x+5) + B(x-3)$$

$$\Rightarrow 1 = Ax + 5A + Bx - 3B$$

$$\Rightarrow 1 = x(A+B) + 1.(5A - 3B)$$

$$\Rightarrow \begin{cases} A + B = 0 \\ 5A - 3B = 1 \end{cases} \Rightarrow \begin{cases} A = -B \\ -5B - 3B = 1 \end{cases}$$

$$\begin{cases} A = -B \\ -8B \ldots \end{cases} \qquad A = \frac{1}{8}.$$

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$$\begin{cases} A + B = 0 \end{cases} \quad \begin{cases} A = -B \\ B - 3B = 1 \end{cases}$$

$$\Rightarrow \begin{cases} A = -B \\ -8B = 1 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{8} \\ B = -\frac{1}{8} \end{cases}$$

write $\frac{1}{(x-3)(x+5)} = \frac{1}{8(x-3)} - \frac{1}{8(x+5)}$.

$$\int \frac{1}{(x-3)(x+5)}\,dx = \int \frac{1}{8(x-3)}\,dx - \int \frac{1}{8(x+5)}\,dx$$

$$= \frac{1}{8}\ln|x-3| - \frac{1}{8}\ln|x+5| + C.$$

b) $\int_{-1}^{0} \frac{x^3 - 4x + 1}{x^2 - 3x + 2}\,dx$ deg top $= 3 > 2$.

We have $x^2 - 3x + 2 = (x-2)(x-1)$

long division

$$\begin{array}{r|l} x^3 - 4x + 1 & x^2 - 3x + 2 \\ -\ \underline{x^3 - 3x^2 + 2x} & x + 3 \\ 3x^2 - 6x + 1 & \\ -\ \underline{3x^2 - 9x + 6} & \\ 3x - 5 & \end{array}$$

$$\Rightarrow \frac{x^3 - 4x + 1}{x^2 - 3x + 2} = x + 3 + \frac{3x - 5}{x^2 - 3x + 2}.$$

$$\begin{array}{r|l} x^3 - 4x + 1 & x^2 - 3x + 2 \\ -\ \underline{x^3 - 3x^2 + 2x} & x + 3 \\ 3x^2 - 6x + 1 & \\ -\ \underline{3x^2 - 9x + 6} & \\ 3x - 5 & \end{array}$$

$(3x - 5)$ remainder

write: $\frac{x^3 - 4x + 1}{x^2 - 3x + 2} = x + 3 + \frac{3x - 5}{x^2 - 3x + 2}$.

$$\int \frac{x^3 - 4x + 1}{x^2 - 3x + 2}\,dx = \int \left(x + 3 + \frac{3x - 5}{x^2 - 3x + 2}\right)dx$$

$$= \int (x+3)\,dx + \int \frac{3x-5}{\ldots}\,dx$$

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$3x - 5$ remainder

$$= x + 3 + \frac{3x-5}{x^2-3x+2}$$

$$\int \frac{x^3-4x+1}{x^2-3x+2}\,dx = \int\left(x+3+\frac{3x-5}{x^2-3x+2}\right)dx$$

$$= \int (x+3)\,dx + \int \frac{3x-5}{x^2-3x+2}\,dx$$

(Easy)

$$\frac{3x-5}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1}$$

$\Rightarrow 3x - 5 = A(x-1) + B(x-2)$

$\Rightarrow 3x - 5 = Ax - A + Bx - 2B$

$\Rightarrow 3x - 5 = x(A+B) + 1(-A-2B)$

$\Rightarrow \begin{cases} A+B=3 \\ -A-2B=-5 \end{cases} \Rightarrow \begin{cases} A = 3-B \\ -(3-B)-2B=-5 \end{cases}$

$\Rightarrow \begin{cases} A = 3-B \\ -3+B-2B=-5 \end{cases} \Rightarrow \begin{cases} A = 3-B \\ -B = -5+3 = -2 \end{cases}$

$\Rightarrow \begin{cases} A = 3-B = 1. \\ B = 2 \end{cases}$

$$\int \frac{3x-5}{(x-2)(x-1)}\,dx = \int\left(\frac{1}{x-2} + \frac{2}{x-1}\right)dx$$